Renormalization of Gauge Theories

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February 2025

Contents

1 Useful identities

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1.1 Feynman integrals

The key to most loop computations is Feynman parametrizations, given by the following integral

$$\frac{1}{A_1^{s_1}\dots A_n^{s_n}} = \frac{\Gamma(s_1+\dots+s_n)}{\Gamma(s_1)\dots\Gamma(s_n)}\int_0^1 dx_1\dots\int_0^1 dx_n \frac{\delta(x_1+\dots+x_n-1)}{(x_1A_1+\dots+x_nA_n)^{s_1+\dots+s_n}} \tag{1.1}$$

which can be derived using Schwinger parameters. Next, let’s compute the tadpole integral

$$
\begin{aligned}
T_{n,j}(m^2) &= \int \frac{d^dk}{(2\pi)^d} \frac{k^j}{(k^2+m^2)^n} \\
&= \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \int_0^\infty dk \frac{k^{j+d-1}}{(k^2+m^2)^n} \\
&= \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} m^{-2n} \int_0^\infty dk k^{j+d-1} \left(1+\frac{k^2}{m^2}\right)^{-n} \\
&= \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} m^{j+d-2n} \int_0^\infty du u^{j+d-1} \left(1+u^2\right)^{-n} \\
&= \frac{\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} m^{j+d-2n} \int_0^1 dt t^{n-\frac{3}{2}-\frac{j+d-1}{2}} (1-t)^{\frac{j+d-1-1}{2}} \\
&= \frac{\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} m^{j+d-2n} \int_0^1 dt t^{n-\frac{j+d}{2}-1} (1-t)^{\frac{j+d}{2}-1} \\
&= \frac{\Gamma\left(n-\frac{j+d}{2}\right)\Gamma\left(\frac{j+d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)(n-1)!} m^{j+d-2n}
\end{aligned} \tag{1.2}
$$

where we've used the following nontrivial substitution:

$$
t = (1+u^2)^{-1}, \quad dt = -\frac{2u}{(1+u^2)^2} du = -2t^{3/2}(1-t)^{1/2} du \tag{1.3}
$$

It's worth mentioning that the previous class of integrals gives us access to the following using isotropy of the integrand:

$$
T_{\mu\nu}(n, m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k_\nu}{(k^2+m^2)^n} = \frac{1}{d} g_{\mu\nu} T_{n,2}(m^2) = \frac{\Gamma\left(n-1-\frac{d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)}{d(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)(n-1)!} m^{d+2-2n} g_{\mu\nu} \tag{1.4}
$$

Vacuum polarization computations also require we evaluate

$$
I = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-m_1^2)[(p+k)^2-m_2^2]}, \quad J^\mu = \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2-m_1^2)[(p+k)^2-m_2^2]}, \quad K^{\mu\nu} = \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2-m_1^2)[(p+k)^2-m_2^2]} \tag{1.5}
$$

$$
\begin{aligned}
I &= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(k+xp)^2 + x(1-x)p^2 + (1-x)m_1^2 + xm_2^2\right]^2} = i \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2+\Delta^2\right]^2} \\
&= i \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \Delta^{d-4} \supset \frac{i}{8\pi^2\varepsilon}
\end{aligned} \tag{1.6}
$$

$$
\begin{aligned}
J^\mu &= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{\left[(k+xp)^2 + x(1-x)p^2 + (1-x)m_1^2 + xm_2^2\right]^2} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu - xp^\mu}{\left[k^2-\Delta^2\right]^2} \\
&= -ip^\mu \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx x \Delta^{d-4} \supset -\frac{ip^\mu}{16\pi^2\varepsilon}
\end{aligned} \tag{1.7}
$$

$$K^{\mu\nu} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{\left[(k+xp)^2 + x(1-x)p^2 + (1-x)m_1^2 + xm_2^2\right]^2} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu + x^2 p^\mu p^\nu}{\left[k^2 - \Delta^2\right]^2}$$
$$= ip^\mu p^\nu \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx x^2 \Delta^{d-4} - \frac{i}{4} g^{\mu\nu} \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(1+\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^1 dx \Delta^{d-2}$$
$$\supset \frac{ip^\mu p^\nu}{24\pi^2\varepsilon} - \frac{ig^{\mu\nu}(p^2 + 3(m_1^2+m_2^2))}{96\pi^2\varepsilon} = \frac{i}{24\pi^2\varepsilon}\left(p^\mu p^\nu - \frac{1}{4}(p^2 + 3m_1^2 + 3m_2^2) g^{\mu\nu}\right) \tag{1.8}$$

1.2 Diracology

There are some Clifford algebraic expressions arising in the renormalization of QED, which stay useful in Yang-Mills theory and are summarized below:

$$\gamma^\mu \gamma_\mu = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = \delta^\mu{}_\mu = d \tag{1.9}$$

$$\gamma^\mu \not{k} \gamma_\mu = k_\nu \gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\not{k} \tag{1.10}$$

$$\mathrm{Tr}\left[\gamma^\mu \not{k} \gamma^\nu \not{p}\right] = 4\left(k^\mu p^\nu + k^\nu p^\mu - g^{\mu\nu} k\cdot p\right) \tag{1.11}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right] = d g^{\mu\nu} \tag{1.12}$$

1.3 Color factors

In non abelian gauge theory, group-theoretic factors involving traces and products of generators appear in front of Feynman integrals. Let τ^a be the generators of the relevant Lie algebra $\mathfrak{g}$, T^a those in the defining representation. It can be shown that (using Schur's lemma among other things):

$$T^a T^a = C_D 1 \tag{1.13}$$

$$\mathrm{Tr}\left(T^a T^b\right) = T_D \delta^{ab} \tag{1.14}$$

In particular, $T_D = 1/2$, since we compare invariant forms to twice that of the form associated to the defining representation. Taking the trace of the Casimir for $\mathfrak{g} = \mathfrak{su}(N)$ yields

$$C_D = \frac{1}{2}\frac{\dim \mathfrak{g}}{\dim \pi} = \frac{N^2-1}{2N} \tag{1.15}$$

As for the adjoint representation,

$$C_A = T_A = N \tag{1.16}$$

We also define structure constants f^{abc} such that

$$[\tau^a, \tau^b] = i f^{abc} \tau^c \tag{1.17}$$

In our computations of vacuum polarization and vertex renormalization, products of structure constants will come up. To this end, let's look at the adjoint representation;

$$\mathrm{ad}_{\tau^a}(\tau^b) = i f^{abc} \tau^c = T_{\mathrm{adj}}(\tau^a)^{bc} \tau^c \tag{1.18}$$

This entails

$$f^{dac} f^{cbd} = f^{dac} f^{dcb} = -T_{\mathrm{adj}}(\tau^d)^{ac} T_{\mathrm{adj}}(\tau^d)^{cb} = -[T^d_{\mathrm{adj}} T^d_{\mathrm{adj}}]_{ab} = -C_A \delta_{ab} \tag{1.19}$$

Next, recall $f^{dac}f^{cbd} = -C_A\delta_{ab}$:

$$if^{bac}T^aT^c = -f^{bac}f^{acd}T^d - if^{bac}T^aT^c = -C_AT^b - f^{bac}T^aT^c \tag{1.20}$$

Thus

$$T^aT^bT^a = C_DT^b + if^{bac}T^aT^c = \left(C_D - \frac{1}{2}C_A\right)T^b \tag{1.21}$$

Lastly, for any complex matrix M:

$$M = \frac{\operatorname{Tr} M}{N}1 + 2\operatorname{Tr}(MT^c)T^c \tag{1.22}$$

i.e.

$$\left(\delta_{ki}\delta_{lj} - \frac{1}{N}\delta_{kl}\delta_{ij} - 2T^c_{lk}T^c_{ij}\right)M_{kl} = 0 \tag{1.23}$$

This being valid for all M, we arrive at

$$T^c_{lk}T^c_{ij} = \frac{1}{2}\delta_{ki}\delta_{lj} - \frac{1}{2N}\delta_{kl}\delta_{ij} \tag{1.24}$$

Multiplying the above by a third generator in the defining representation:

$$[T^cT^bT^c]_{li} = T^c_{lk}T^c_{ki}T^c_{ij} = -\frac{1}{2N}T^b_{li} \tag{1.25}$$

and comparing with a previous computation

$$T^aT^bT^a = C_DT^b - \frac{1}{2}C_AT^b = -\frac{1}{2N}T^b \tag{1.26}$$

Hence $C_A = N$.

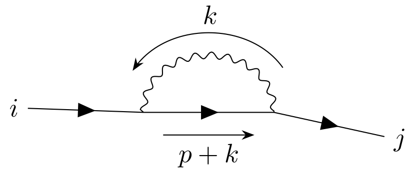
2 Renormalization of Quantum Electrodynamics

Ignoring the gauge fixing part of the Lagrangian with counter-terms:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\left(\partial_\mu[A_\nu] - \partial_\nu[A_\mu]\right)^2 + [\bar\psi]\left(i\partial\!\!\!/ - m\right)[\psi] - e[\bar\psi]\gamma^\mu[\psi] \\ &- \frac{1}{4}\delta_3\left(\partial_\mu[A_\nu] - \partial_\nu[A_\mu]\right)^2 + i\delta_2[\bar\psi]\partial\!\!\!/[\psi] - m\delta_m[\bar\psi][\psi] - e\delta_1[\bar\psi]\gamma^\mu[\psi]\end{aligned} \tag{2.1}$$

We need to figure out the three-point vertex for the gluon.

2.1 Electron self-energy



$$
\begin{aligned}
&= \int \frac{d^dk}{(2\pi)^d}(-ie\gamma^\mu)\frac{i(\not{k}+m)}{k^2-m^2}(-ie\gamma_\mu)\frac{-i}{(k-p)^2} \\
&= -e^2\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2}\frac{\gamma^\mu\not{k}\gamma_\mu+m\gamma^\mu\gamma_\mu}{k^2-m^2} \\
&= -e^2\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2}\frac{(2-d)\not{k}+dm}{k^2-m^2} \\
&\supset -ie^2\frac{1}{8\pi^2\varepsilon}\left(4m-\not{p}\right)
\end{aligned}
\tag{2.2}
$$

The self-energy's singular part at one loop is:

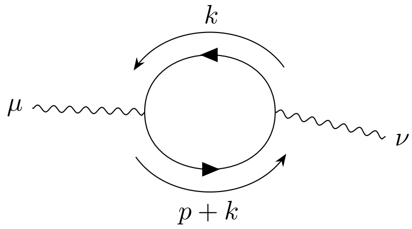
$$
i\Sigma = -ie^2\frac{1}{8\pi^2\varepsilon}\left(4m-\not{p}\right)+i\left(\delta_2\not{p}-m(\delta_m+\delta_2)\right) \tag{2.3}
$$

Thus the renormalization conditions give:

$$
\delta_2 = -\frac{e^2}{8\pi^2\varepsilon}, \quad \delta_m = -\frac{3e^2}{8\pi^2\varepsilon} \tag{2.4}
$$

2.2 Vacuum polarization

Don't forget extra minus from fermion loop (Wick theorem)!



$$
\begin{aligned}
&= -\int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}\left[(-ie\gamma^\mu)\frac{i(\not{k}+m)}{k^2-m^2}(-ie\gamma^\nu)\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right] \\
&= -e^2\int \frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\right]+\mathrm{Tr}\left[\gamma^\mu\not{k}\gamma^\nu\not{p}\right]+m^2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]}{(k^2-m^2)((k+p)^2-m^2)} \\
&= -e^2\int \frac{d^dk}{(2\pi)^d}\frac{dm^2g^{\mu\nu}+8k^\mu k^\nu+8k^{(\mu}p^{\nu)}-4k^2g^{\mu\nu}-4p\cdot kg^{\mu\nu}}{(k^2-m^2)((k+p)^2-m^2)} \\
&\supset -i\frac{e^2}{8\pi^2\varepsilon}\left(4m^2g^{\mu\nu}-\frac{2}{3}(2p^\mu p^\nu-p^2g^{\mu\nu})\right)+i\frac{e^2}{2\pi^2\varepsilon}g^{\mu\nu}\left(\frac{1}{6}p^2-m^2\right) \\
&= i\frac{e^2}{6\pi^2\varepsilon}\left(p^\mu p^\nu-p^2g^{\mu\nu}\right)
\end{aligned}
\tag{2.5}
$$

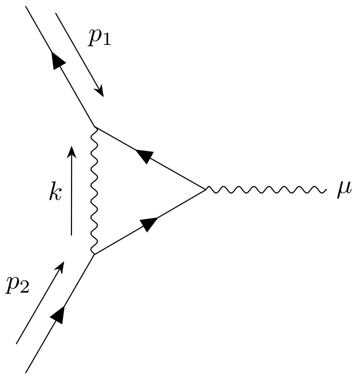
Hence the singular part of the two-point function of the photonin the renormalized theory is

$$
i\Pi = i\frac{e^2}{6\pi^2\varepsilon}\left(p^\mu p^\nu-p^2g^{\mu\nu}\right)-i\delta_3(p^2g^{\mu\nu}-p^\mu p^\nu) \tag{2.6}
$$

i.e.

$$\delta_3 = -\frac{e^2}{6\pi^2\varepsilon} \tag{2.7}$$

2.3 Renormalization of the vertex



(2.8)

$$\begin{aligned}
&= \int \frac{d^dk}{(2\pi)^d}(-ie\gamma^\mu)\frac{i(\not{p}_1+\not{k}+m)}{(p_1+k)^2-m^2}(-ie\gamma^\nu)\frac{i(\not{p}_2-\not{k}+m)}{(p_2-k)^2-m^2}(-ie\gamma_\mu)\frac{-i}{k^2} \\
&= -e^3\int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\mu(\not{p}_1+\not{k}+m)\gamma^\nu(\not{p}_2-\not{k}+m)\gamma_\mu}{k^2[(p_1+k)^2-m^2][(p_2-k)^2-m^2]}
\end{aligned}$$

This integral is very tough to compute, albeit doable. Since our main concern is UV divergences, it suffices to consider the $k \to \infty$ behavior of the integrand.

$$D_3 \sim e^3 \int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\alpha \not{k}\gamma^\mu \not{k}\gamma_\alpha}{k^6} \tag{2.9}$$

Using some straightforward Clifford algebra with $d \to 4$:

$$\begin{aligned}
\gamma^\alpha \not{k}\gamma^\mu \not{k}\gamma_\alpha &= k_\nu k_\rho \gamma^\alpha\gamma^\nu\gamma^\mu\gamma^\rho\gamma_\alpha \\
&= 2k^2\gamma^\mu - 4k^\mu \not{k}
\end{aligned} \tag{2.10}$$

It's worth noting (via some Passarino-Veltman reduction):

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^\mu \not{k}}{k^6} = \frac{1}{d}\gamma^\mu \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2} \tag{2.11}$$

Hence, taking care not to forget a Wick rotation:

$$D_3 \sim -\frac{ie^3\gamma^\mu 2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)(2\pi)^d}\int \frac{dk}{k^{1+\varepsilon}} \sim -i\gamma^\mu \frac{e^3}{8\pi^2\varepsilon} \tag{2.12}$$

The counter term being $-ie\gamma^\mu\delta_1$, this leads to:

$$\delta_1 = -\frac{e^2}{8\pi^2\varepsilon} = \delta_2 \tag{2.13}$$

where we incidentally recover a famous consequence of Ward-Takahashi.

2.4 β-function

By definition:

$$Z_e = \frac{Z_1}{Z_2\sqrt{Z_3}} = 1 - \frac{1}{2}\delta_3 = 1 + \frac{e^2}{12\pi^2\varepsilon} \tag{2.14}$$

$[e] + d - 1 + (d-2)/2 = d$ i.e. $[e] = 2 - \frac{d}{2}$. Hence

$$e_0 = \left(e + \frac{e^3}{12\pi^2\varepsilon}\right)\mu^{\varepsilon/2} \tag{2.15}$$

$$\beta_e = -\frac{\varepsilon e}{2} + \frac{e^3}{12\pi^2} + O(e^4) \tag{2.16}$$

The relevant dimensionless parameter for QED is the fine structure constant $\alpha = e^2/(4\pi)$, and the corresponding β-function at $d = 4$ is

$$\beta_\alpha = \frac{2\alpha^2}{3\pi} > 0 \tag{2.17}$$

Therefore, there exists a high energy scale Λ_{QED} at which α diverges, such that

$$\alpha(\mu) = \frac{3\pi}{2\ln\left(\frac{\Lambda_{\mathrm{QED}}}{\mu}\right)} \tag{2.18}$$

The theory is therefore free at low energies, and strongly interacting at high energies, where perturbation theory breaks down. This is referred to as the Landau pole, however experiment suggests it is of the order of 10^{286} and therefore completely out of reach. To estimate this value, take $\mu = m_e$ and e the usual value in classical electrodynamics.

$$\Lambda_{\mathrm{QED}} = m_e \exp\left(137 \times \frac{3\pi}{2}\right) \tag{2.19}$$

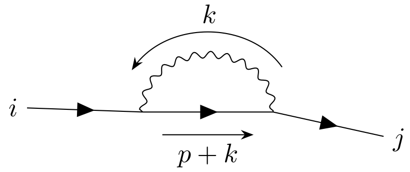
3 Renormalization of Yang-Mills theory

Let's write down the full bare Yang-Mills Lagrangian.

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\partial_\mu A^a_\nu \partial^\nu A^{a,\mu} - \partial_\mu A^a_\nu \partial^\mu A^{a,\nu}\right) + \bar{\psi}(i\not\partial - m)\bar{\psi} - \frac{1}{2\xi}\left(\partial_\mu A^\mu_a\right)^2 + \partial_\mu \bar{c}_a \partial^\mu c_a \\ &+ g A^a_\mu \bar{\psi}\gamma^\mu T^a \psi + g f_{abc} A^b_\mu \partial^\mu \bar{c}_a c_c - g\partial_\mu A^a_\nu f^{abc} A^{b,\mu} A^{c,\nu} - \frac{g^2}{4} f^{abe} f^{cde} A^a_\mu A^b_\nu A^{c,\mu} A^{a,\nu}\end{aligned} \tag{3.1}$$

3.1 Fermion self-energy

The fermion self-energy will be almost identical to that of QED, up to a Lie algebra factor and color indices.



$$\begin{aligned}&= \left[\int \frac{d^d k}{(2\pi)^d} ig\gamma^\mu T^a \frac{i(\not k + m)}{k^2 - m^2} ig\gamma_\mu T^a \frac{-i}{(k-p)^2}\right]_{ij} = C(\pi)\delta_{ij} \times \text{QED integral} \\ &\supset -ig^2 C_D \delta_{ij} \frac{1}{8\pi^2\varepsilon}(4m - \not p)\end{aligned} \tag{3.2}$$

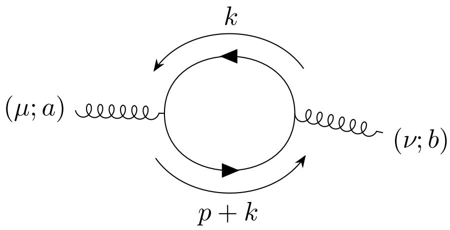
The counter-term being $i(\delta_2 \not p - m(\delta_2 + \delta_m + \delta_2\delta_m))$, we get

$$\delta_2 = -\frac{C_D g^2}{8\pi^2\varepsilon}, \quad \delta_m = -\frac{3C_D g^2}{8\pi^2\varepsilon} \tag{3.3}$$

3.2 Vacuum polarization

3.2.1 Fermion loop

The first one involving fermions is easy in light of our previous computations of QED:

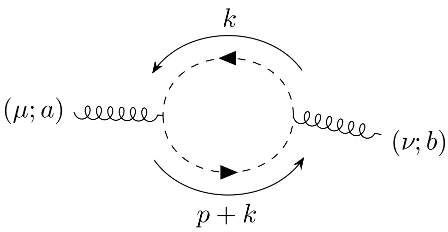


(3.4)

$$
\begin{aligned}
&= -\int \frac{d^d k}{(2\pi)^d} \operatorname{Tr}\left[(ig\gamma^\mu T^a)\frac{i(\not{k}+m)}{k^2-m^2}(ig\gamma^\nu T^b)\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right] \\
&= \operatorname{Tr}\left(T^a T^b\right) \times \text{QED integral} \\
&= iT_D\delta^{ab}\frac{g^2}{6\pi^2\varepsilon}\left(p^\mu p^\nu - p^2 g^{\mu\nu}\right)
\end{aligned}
$$

3.2.2 Ghost loop

Let's consider the ghost contribution to the gluon two-point function. $-gf_{abc}p_\mu$.



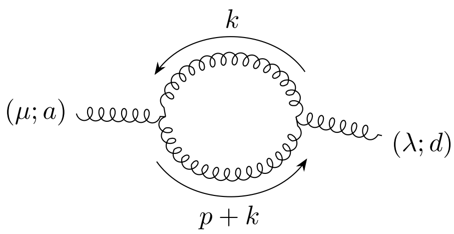
(3.5)

$$
\begin{aligned}
&= -\int \frac{d^d k}{(2\pi)^d}(-gf_{cbd}k^\mu)\frac{i}{k^2}(-gf_{dac}(p+k)^\nu)\frac{i}{(p+k)^2} \\
&= g^2 f_{dac}f_{cbd}\int \frac{d^d k}{(2\pi)^d}\frac{k^\mu k^\nu + k^\mu p^\nu}{k^2(p+k)^2} := D
\end{aligned}
$$

Hence using the integrals from the first section:

$$
D \supset \frac{ig^2 C_A}{8\pi^2\varepsilon}\delta_{ab}\left(\frac{1}{12}g^{\mu\nu}p^2 + \frac{1}{6}p^\mu p^\nu\right) \tag{3.6}
$$

3.2.3 Gluon loop



$$
\begin{aligned}
&= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (-g f_{abc}) \left(g_{\mu\rho}(p-k)_\nu + g_{\mu\nu}(-2p-k)_\rho + g_{\nu\rho}(2k+p)_\mu \right) \frac{-i}{k^2} \frac{-i}{(p+k)^2} \\
&\times (g f_{dbc}) \left(\delta^\rho_\lambda (p-k)^\nu + \delta^\nu_\lambda (-2p-k)^\rho + g^{\nu\rho}(2k+p)_\lambda \right) \\
&= \frac{1}{2} g^2 f_{abc} f_{dbc} \int \frac{d^d k}{(2\pi)^d} \frac{N_{\mu\lambda}}{k^2 (p+k)^2} \\
&= \frac{1}{2} g^2 C_A \delta_{ad} \int \frac{d^d k}{(2\pi)^d} \frac{N_{\mu\lambda}}{k^2 (p+k)^2} =: D
\end{aligned}
\tag{3.7}
$$

where we define

$$
\begin{aligned}
N_{\mu\lambda} &= g_{\mu\lambda}(p-k)^2 - (p-k)_\lambda (2p+k)_\mu + (p-k)_\mu (2k+p)_\lambda - (p-k)_\mu (2p+k)_\lambda \\
&+ g_{\mu\lambda}(2p+k)^2 - (2p+k)_\mu (2k+p)_\lambda + (p-k)_\lambda (2k+p)_\mu - (2p+k)_\lambda (2k+p)_\mu \\
&+ d(2k+p)_\mu (2k+p)_\lambda \\
&= g_{\mu\lambda}(p-k)^2 - 2(p-k)_{(\lambda}(2p+k)_{\mu)} + 2(p-k)_{(\lambda}(2k+p)_{\mu)} + g_{\mu\lambda}(2p+k)^2 \\
&- 2(2p+k)_{(\mu}(2k+p)_{\lambda)} + d(2k+p)_\mu (2k+p)_\lambda \\
&= 5p^2 g_{\mu\lambda} + 2g_{\mu\lambda}k^2 + 2g_{\mu\lambda}pk + (d-6)p_\mu p_\lambda + (4d-6)k_\mu k_\lambda + (4d-6)k_{(\mu}p_{\lambda)}
\end{aligned}
\tag{3.8}
$$

Using the 'useful integrals' from the first section, we can now easily compute the corresponding pole in ε:

$$
\begin{aligned}
D &\supset \frac{1}{2} g^2 C_A \delta_{ad} \left[\frac{i}{8\pi^2\varepsilon}(5p^2 g_{\mu\lambda} - 2p_\mu p_\lambda) - \frac{i}{8\pi^2\varepsilon} g_{\mu\lambda} p^2 + \frac{5i}{12\pi^2\varepsilon} \left(p_\mu p_\lambda - \frac{1}{4} p^2 g_{\mu\lambda} \right) - \frac{5i}{8\pi^2\varepsilon} p_\mu p_\lambda \right] \\
&\supset \frac{ig^2}{16\pi^2\varepsilon} C_A \delta_{ad} \left(\frac{19}{6} p^2 g_{\mu\lambda} - \frac{11}{3} k_\mu k_\lambda \right)
\end{aligned}
\tag{3.9}
$$

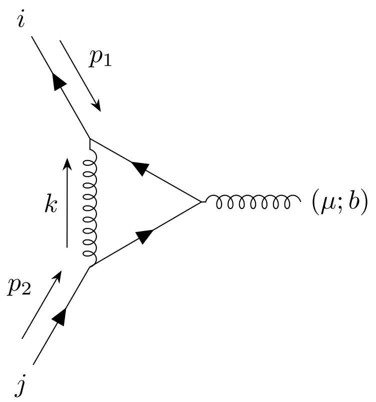
3.2.4 Full counter-term

If we include n_F flavors of quarks, we arrvive at the following vertex counter-term:

$$
\delta_3 = \frac{g^2}{16\pi^2\varepsilon} \left(\frac{10}{3} C_A - \frac{8}{3} n_f T_D \right) \tag{3.10}
$$

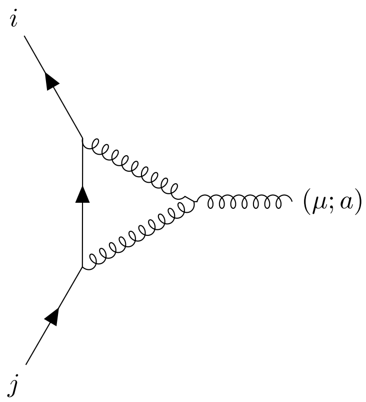
3.3 Renormalization of the vertex

3.3.1 Fermion half-loop



$$
\begin{aligned}
&= \int \frac{d^d k}{(2\pi)^d} (ig\gamma^\mu T^a) \frac{i(\not{p}_1 + \not{k} + m)}{(p_1+k)^2 - m^2} (ig\gamma^\nu T^b) \frac{i(\not{p}_2 - \not{k} + m)}{(p_2-k)^2 - m^2} (ig\gamma_\mu T^a) \frac{-i}{k^2} \\
&= T^a T^b T^a \times \text{QED integral} \\
&\supset \left(C_D - \frac{1}{2} C_A \right) iT^b \gamma^\mu \frac{g^3}{8\pi^2 \varepsilon}
\end{aligned}
\tag{3.11}
$$

3.3.2 Gluon half-loop



$$
\begin{aligned}
&= ig^3 f^{abc} T^c T^b \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\rho (\not{k} + m) \gamma^\nu}{k^2 - m^2} \left[g_{\mu\nu} (q_1 + 2p_2 + k)_\rho + g_{\nu\rho} (q_1 - q_2 - 2k)_\mu + g_{\rho\mu} (-2p_1 - p_2 + k)_\nu \right] \\
&\times \frac{1}{(q_1 - k)^2 (q_2 + k)^2} \\
&\supset \frac{1}{2} g^3 C_A T^a \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\rho \not{k} \gamma^\nu}{k^6} \left(g_{\mu\nu} k_\rho - 2 g_{\nu\rho} k_\mu + g_{\rho\mu} k_\nu \right) \\
&= \frac{1}{2} g^3 C_A T^a \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^6} \left(\not{k}\not{k}\gamma_\mu - 2\gamma^\rho \not{k} \gamma_\rho k_\mu + \gamma_\mu \not{k}\not{k} \right) \\
&= \frac{g^3 C_A T^a}{(2\pi)^4} \int \frac{d^d k}{k^6} \left(k^2 \gamma_\mu + 2\not{k} k_\mu \right) \\
&= \frac{3 g^3 C_A T^a}{2(2\pi)^4} i\gamma_\mu \int \frac{d^d k}{k^4} = \frac{3g^3}{16\pi^2} i C_A T^a \gamma_\mu \frac{1}{\varepsilon}
\end{aligned}
\tag{3.12}
$$

where we've added a factor of i accounting for Wick rotation. The counter-term is $ig\gamma^\mu T^a \delta_1$, hence:

$$\delta_1 = -\left(C_D - \frac{1}{2}C_A\right)\frac{g^2}{8\pi^2\varepsilon} - \frac{3C_A g^2}{16\pi^2\varepsilon}$$
$$= -(C_D + C_A)\frac{g^2}{8\pi^2\varepsilon} \tag{3.13}$$

3.4 β-function

$$\delta_2 = -\frac{C_D g^2}{8\pi^2\varepsilon}, \quad \delta_m = -\frac{3C_D g^2}{8\pi^2\varepsilon} \tag{3.14}$$

$$\delta_3 = \frac{g^2}{16\pi^2\varepsilon}\left(\frac{10}{3}C_A - \frac{8}{3}n_f T_D\right) \tag{3.15}$$

$$\delta_1 = -(C_D + C_A)\frac{g^2}{8\pi^2\varepsilon} \tag{3.16}$$

By definition:

$$Z_g = \frac{Z_1}{Z_2\sqrt{Z_3}} \tag{3.17}$$

i.e.

$$\delta_g = \delta_1 - \delta_2 - \frac{1}{2}\delta_3 = -\frac{g^2}{16\pi^2\varepsilon}\left(\frac{11}{3}C_A - \frac{4}{3}n_f T_D\right) \tag{3.18}$$

which leads to the following β-function:

$$\beta_g = -\frac{\varepsilon}{2}g - \frac{g^3}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}n_f T_D\right] \tag{3.19}$$

3.5 Quantum Chromodynamics

For QCD/$\mathfrak{su}(3)$, $C_A = 3$, $T_D = 1/2$, $n_f = 6$.

$$\beta_g = -\frac{7g^3}{16\pi^2} \tag{3.20}$$

i.e.

$$\beta_\alpha = -\frac{7g^4}{32\pi^3} = -\frac{7}{2\pi}\alpha^2 \tag{3.21}$$

Hence

$$\alpha(\mu) = \frac{2\pi}{7\ln\left(\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)} \tag{3.22}$$

The theory is asymptotically free and at low energies strongly interacting, suggesting confinement. Note $\Lambda_{\mathrm{QCD}} = 200 - 300$ MeV. This is a physical scale, at which the theory is non-perturbative and one expects confinement. Let's derive this estimate. Let's take μ to be the mass m of one of the quarks, say the top quark, and α of the order of 1. Then cleary Λ_{QCD} is of the order of 100 Mev.

$$\tag{3.23}$$